THE CHINESE UNIVERSITY OF HONG KONG
MATH 2230 (Second Term, 2023-24)
Complex Variables with Applications
Midterm 1
Time: 7-8 p.m, 27th February 2024

Answer all FIVE questions.

1. (20 pts) Simplify the expression $\frac{(\sqrt{3}+i)^9}{(\sqrt{2}+\sqrt{2}i)^2}$ and express the answer in the form $x + iy$.

2. (20 pts) Simplify the expression $i^{(1+i)}$ where the power function is defined on the principal branch $(-\pi < \operatorname{Arg} z < \pi)$ and express the answer in polar form.

3. (20 pts) Given the map $f(z) = \frac{z-i}{z+i}$,

   (a) Find the image of the real axis $(y = 0)$ under f.

   (b) What is the image of the upper half plane $(y \geq 0)$?

4. (20 pts) Let C be the positively oriented circle $|z| = 4$

   (a) Compute $\int_C x\,dz$, where $x = Re(z)$.

   (b) Compute $\int_C \frac{1}{(z+1)^n}\,dz$, where $n > 0$ is a positive integer.

5. (20 pts) Show that $f(z) = (z+1)^2 - 3\bar{z}$ is nowhere complex differentiable.

Math 2230 Midterm 1

Q1: $\sqrt{3}+i = 2e^{\frac{\pi}{6}i}$ $\quad \sqrt{2}+\sqrt{2}i = 2e^{\frac{\pi}{4}i}$

$\therefore \dfrac{(\sqrt{3}+i)^9}{(\sqrt{2}+\sqrt{2}i)^2} = \dfrac{2^9 e^{\frac{3}{2}\pi i}}{2^2 e^{\frac{\pi}{2}i}} = 2^7 e^{\pi i} = -128$

Q2: $i^{(1+i)} = \exp[(1+i)\,\mathrm{Log}\, i]$

$= \exp[(1+i)(\ln 1 + \frac{\pi}{2}i)]$

$= \exp(\frac{\pi}{2}i - \frac{\pi}{2})$

$= e^{-\frac{\pi}{2}} i$

Q3: $w = f(z) = \dfrac{z-i}{z+i} = 1-2i\cdot\dfrac{1}{z+i}$; $z: \{Im(z) \geq 0\}$ — for (b)

(a) is simple boundary

let $z_1 = z+i$, $w = 1-2i\cdot\frac{1}{z_1}$; $z_1: \{Im(z_1) \geq 1\}$

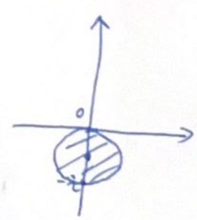
let $z_2 = \frac{1}{z_1}$, $w = 1-2i\cdot z_2$; $z_2: \{Im(\frac{1}{z_2}) \geq 1\}$ $\quad$ let $z_2 = x_2 + iy_2$

$= \left\{ Im\left(\dfrac{x_2}{x_2^2+y_2^2} - i\dfrac{y_2}{x_2^2+y_2^2}\right) \geq 1 \right\}$

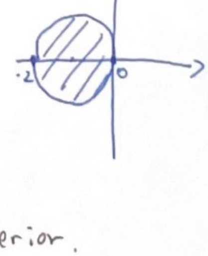
$= \left\{ -\dfrac{y_2}{x_2^2+y_2^2} \geq 1 \right\}$

$= \left\{ x_2^2+y_2^2+y_2+\frac{1}{4} \leq \frac{1}{4} \right\}$

$= \left\{ x_2^2+(y_2+\frac{1}{2})^2 \leq (\frac{1}{2})^2 \right\} \Leftrightarrow$



let $z_3 = -2i\cdot z_2$, $w = 1+z_3$: z_3 :



w: Unit circle and its interior.

Remark for Q3:

① since ∞ is not often consider as a component of $\mathbb{C}$

(1, 0) should be excluded from the image of $f(z)$

② $|f(z)| = \frac{|z-i|}{|z+i|} \leq 1$ is only a necessary condition for the image

to be unit circle. you still need to clarify why this mapping is an

onto mapping.

Q4: a) $\int_C x\,dz = \frac{1}{2}\int_C z+\bar{z}\,dz$ by cauchy's theorem

$$= \frac{1}{2}\int_C \bar{z}\,dz \qquad z = 4e^{i\theta}$$

$$= \frac{1}{2}\int_0^{2\pi} 16i\,d\theta = 16\pi i$$

b) let $f(z) \equiv 1$ Then $f(z)$ is an entire function, by cauchy's integral formula.

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \int_C \frac{f(z)}{(z-z_0)^n}\,dz$$

so when $n=1$, $\int_C \frac{f(z)}{z+1}\,dz = \int_C \frac{1}{z+1}\,dz = f(-1)\cdot 2\pi i = 2\pi i$

when $n>1$, $\int_C \frac{1}{(z+1)^n}\,dz = f^{(n)}(-1)\cdot\frac{2\pi i}{n!} = 0$ since $f^{(n)}(-1) = 0$

Q5: $f(x+iy) = (x^2-x-y^2+1) + (2xy+5y)i$

$\therefore \frac{\partial u}{\partial x} = 2x-1 \qquad \frac{\partial v}{\partial y} = 2x+5$

$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ for all points on $\mathbb{C}$ $\Rightarrow$ f is nowhere differentiable